BC 3
1994

A particle moves along the graph of $y = \cos x$ so that the x-component of acceleration is always 2. At time $t = 0$, the particle is at the point $(\pi, -1)$ and the velocity vector of the particle is $(0,0)$.

(a) Find the x- and y- coordinates of the position of the particle in terms of t.

(b) Find the speed of the particle when its position is $(4, \cos 4)$.

(a) $x''(t) = 2 \Rightarrow x'(t) = 2t + C$

$x'(0) = 0 \Rightarrow C = 0$; $x'(t) = 2t$

$x(t) = t^2 + k$, $x(0) = \pi = k$

$x(t) = t^2 + \pi$

$y(t) = \cos(t^2 + \pi)$

5 { 1: $\frac{dx}{dt} = 2t + C$ (with or without constant)
1: $C = 0$
1: $x(t) = t^2 + K$ (with or without constant)
1: $K = \pi$
1: $y(t) = \cos(x(t))$ }

b) $\frac{dy}{dt} = -2t\sin(t^2 + \pi)$

$s(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{(2t)^2 + (-2t\sin(t^2+\pi))^2}$

$= \sqrt{4t^2 + 4t^2\sin^2(t^2+\pi)}$

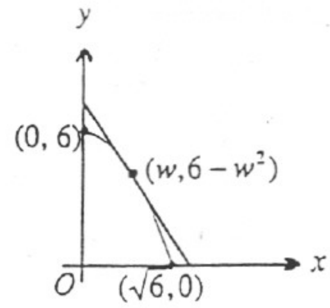
WHEN $x = 4$, $t^2 + \pi = 4$; $t^2 = 4 - \pi$

$s = \sqrt{4(4-\pi) + 4(4-\pi)\sin^2 4}$

$\doteq 2.324$

4 { 1: $\frac{dy}{dt}$ (y must be of form $\cos(f(t))$)
1: Solves $x(t) = 4$ for t or t^2
1: Finds $\frac{dx}{dt}$ and $\frac{dy}{dt}$ when $x=4$
<-1>: Numerical errors in computing speed
1: Finds speed }

BC 4
1994



Note: Figure not to scale.

Let $f(x) = 6 - x^2$. For $0 < w < \sqrt{6}$, let $A(w)$ be the area of the triangle formed by the coordinate axes and the line tangent to the graph of f at the point $(w,\ 6 - w^2)$. (See figure above.)

(a) Find $A(1)$.

(b) For what value of w is $A(w)$ a minimum?

(a) $f(x) = 6 - x^2;\ \ f'(x) = -2x$

$$f'(1) = -2$$

$y - 5 = -2(x - 1)$ or $y = -2x + 7$

x int: $\dfrac{7}{2}$ $\qquad$ y int: 7

$$A(1) = \frac{1}{2}\left(\frac{7}{2}\right)(7) = \frac{49}{4}$$

$$3\begin{cases} 1\text{: } f'(1) = -2 \\ 1\text{: Finds equation of line or } x \text{ intercept or } y \text{ intercept} \\ 1\text{: Answer} \end{cases}$$

(b) $f'(w) = -2w;\quad y - (6 - w^2) = -2w(x - w)$

x int: $\dfrac{6 + w^2}{2w}$ $\qquad$ y int: $6 + w^2$

$$A(w) = \frac{(6 + w^2)^2}{4w}$$

$$A'(w) = \frac{4w\left(2(6 + w^2)(2w)\right) - 4(6 + w^2)^2}{16w^2}$$

$A'(w) = 0$ when $(6 + w^2)(3w^2 - 6) = 0$

$$w = \sqrt{2}$$

A': $-$ on $(0, \sqrt{2})$, $+$ on $(\sqrt{2}, \sqrt{6})$; axis w marked at 0, $\sqrt{2}$, $\sqrt{6}$

$$6\begin{cases} 1\text{: Equation of line} \\ 1\text{: Expresses } x \text{ intercept or } y \text{ intercept in terms of } w. \\ 1\text{: } A(w) \\ 1\text{: } A'(w) \\ 1\text{: Solves } A'(w) = 0 \\ 1\text{: Shows solution yields a minimum.} \end{cases}$$

Note: $A(w)$ must be of the form $\dfrac{P(w)}{Q(w)}$, neither constant, to be eligible for derivative point.

BC 5
1994

Let f be the function given by $f(x) = e^{-2x^2}$.

(a) Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x = 0$.

(b) Find the interval of convergence of the power series for $f(x)$ about $x = 0$. Show the analysis that leads to your conclusion.

(c) Let g be the function given by the sum of the first four nonzero terms of the power series for $f(x)$ about $x = 0$. Show that $|f(x) - g(x)| < 0.02$ for $-0.6 \le x \le 0.6$.

(a) $e^{u} = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \cdots + \frac{u^n}{n!} + \cdots$

$e^{-2x^2} = 1 - 2x^2 + \frac{4x^4}{2} - \frac{8x^6}{3!} + \cdots + \frac{(-1)^n 2^n x^{2n}}{n!} + \cdots$

3 { 1: 1st and 2nd terms
1: 3rd and 4th terms
1: general term }

(b) The series for e^u converges for $-\infty < u < \infty$

So the series for e^{-2x^2} converges for $-\infty < -2x^2 < \infty$

and, thus, for $-\infty < x < \infty$

OR

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^{n+1}x^{2(n+1)}}{(n+1)!}\cdot\frac{n!}{(-1)^n 2^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\frac{2}{n+1}x^2 = 0 < 1$$

So the series for e^{-2x^2} converges for $-\infty < x < \infty$

3 { 1: interval for e^u series
1: makes connection
1: answer

OR

1: sets up ratio
1: computes limit
1: answer }

(c) $f(x) - g(x) = \frac{16x^8}{4!} - \frac{32x^{16}}{5!} + \cdots$

This is an alternating series for each x, since powers of x are even.

Also, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{n+1}x^2 < 1$ for $-0.6 \le x \le 0.6$

so terms are decreasing in absolute value.

Thus $|f(x) - g(x)| \le \frac{16x^8}{4!} \le \frac{16(.6)^8}{4!}$

$= 0.011\cdots < 0.02$

3 { 1: alternating series bound of $\frac{16x^8}{4!}$
1: uses $x = 0.6$
1: Conclusion relates error at x to error bound at 0.6 }

BC 6
1994

Let f and g be functions that are differentiable for all real numbers x and that have the following properties.

(i) $f'(x) = f(x) - g(x)$

(ii) $g'(x) = g(x) - f(x)$

(iii) $f(0) = 5$

(iv) $g(0) = 1$

(a) Prove that $f(x) + g(x) = 6$ for all x.

(b) Find $f(x)$ and $g(x)$. Show your work.

(a) $f'(x)+g'(x) = f(x)-g(x)+g(x)-f(x) = 0$

so $f+g$ is constant.

$f(0)+g(0)=6$, so $f(x)+g(x)=6$

3 $\begin{cases} 1: f'+g'=0 \\ 1: \therefore f+g \text{ is constant} \\ 1: \text{uses } (f+g)(0) \end{cases}$

(b) $f(x) = 6-g(x)$ so

$g'(x) = g(x)-6+g(x) = 2g(x)-6$

$\frac{dy}{dx} = 2y-6$; $\frac{dy}{2y-6} = dx$

$\frac{1}{2}\ln|2y-6| = x + C$

$\ln|2y-6| = 2x+K$

$|2y-6| = e^{2x+K}$

$2y-6 = Ae^{2x}$

$x=0 \Rightarrow y=1$ so $-4=A$

$2y = -4e^{2x}+6$

$y = 3-2e^{2x} = g(x)$

$f(x) = 6-g(x) = 3+2e^{2x}$

6 $\begin{cases} 1: \text{Differential equation for one function} \\ 1: \text{Separates variables} \\ 1: \text{Antidifferentiation} \\ 1: \text{Evaluates constant} \\ 1: \text{Finds } g \\ 1: \text{Finds } f \end{cases}$

NOTE: MAX 1/6 if not working with DE for one function